Math 352 Real Analysis Exam 2

Instructions: Submit your work on any 5.

1. Let $f: \mathbb{R} \to \mathbb{R}$ be uniformly continuous and define $f_n(x) = f\left(x + \frac{1}{n}\right)$. Show that $f_n \rightrightarrows f$. [20 pts]

2. Let (X, d) and (Y, p) be metric spaces, and let $f, f_n: X \to Y$ with $f_n \rightrightarrows f$ on X. Show that $D(f) \subset \bigcup_{n=1}^{\infty} D(f_n)$ where $D(f)$ is the set of discontinuities of f. [20 pts]

3. Show that $\sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n}$ converges for all $|x| \leq 1$, but that the convergence is not uniform [Hint: Find the sum]. [20 pts]

4. Let $\alpha > 0$. Define $Lip_K\alpha$ be the set of functions $f: [0,1] \to \mathbb{R}$ for which $|f(x) - f(y)| \leq K|x - y|^{\alpha}$ for all x, y. Show that $\{f \in Lip_K\alpha : f(0) = 0\}$ is a compact subset of $C[0,1]$. [20 pts]

5. Define $T: C[a, b] \to C[a, b]$ by

$$T(f)(x) = \int_a^x f(t)dt$$

Show that T maps bounded sets into equicontinuous sets. [20 pts]

6. Give an example of a sequence of functions that is pointwise bounded, but not uniformly bounded. [20 pts]

7. Suppose that $f_n: [a, b] \to \mathbb{R}$ is a sequence of differentiable functions satisfying $|f_n(x)| \leq 1$ for all n and x. Prove that some subsequence of $\{f_n\}$ is uniformly convergent. [20 pts]

8. If f has a bounded derivative on [a, b], show that $\bigvee_a^b f \leq \|f'\|_{\infty}(b - a)$ [20 pts]

9. If $f \in BV[a, b]$ and $[c, d] \subset [a, b]$, show that $f \in BV[c, d]$ and $\bigvee_c^d f \leq \bigvee_a^b f$ [20 pts]

10. Given a sequence of scalars $\{c_n\}$ and a sequence of distinct points $\{x_n\}$ in (a, b), define $f(x) = c_n$ if $x = x_n$ for some n, and $f(x) = 0$ otherwise. Under what condition(s) is f of bounded variation on $[a, b]$? [20 pts]

11. Let X be a compact metric space and let $\{f_n\}$ be an equicontinuous sequence in $C(X)$. Show that $C = \{x \in X: \{f_n(x)\}\ converges\}$ is a closed set in X. [20 pts]